Unit _ III

Unit-3 Continuity

Definition:

Let (M_1, d_1) and (M_2, d_2) be metric spaces

Let $M_1 \to M_2$ be a function. Let $a \in M_1$ and $l \in M_2$. The function f is said to have l as limit as $x \to a$ if given $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < d_1(x, a) < \delta \Rightarrow d_2(f(x), l) < \varepsilon$.

We write $\lim_{x \to a} f(x) = l$.

Definition:

Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Let $a \in M_1$. A function $f: M_1 \to M_2$ is said to be continuous at a if given $\varepsilon > 0$ there exists $\delta > 0$ such that $d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \varepsilon$.

f is said to be continuous if it is continuous at every point of M_1.

Note:

1. f is continuous at a iff
$$\lim_{x \to a} f(x) = f(a)$$
2. The condition $d_1(x, a) < \delta \Rightarrow$
$d_2(f(x), f(a)) < \epsilon$ can be rewritten or
   i) $x \in B(a, \delta) \Rightarrow f(x) \in B(f(a), \epsilon)$
   ii) $f(B(a, \delta)) \subseteq B(f(a), \epsilon)$

Example. 1

Let $f: M_1 \to M_2$ be given by $f(x) = a$ where $a \in M_2$ fixed element.

Let $x \in M_1$ and $\epsilon > 0$ be given
Then for any $\delta > 0$, $f(B(x, \delta)) = \{a\}$ $\subseteq B(a, \epsilon)$
$\therefore$ f is continuous at x.
Since $x \in M_1$ is arbitrary f is continuous.

Example: 2

Let (M_1, d_1) be a discrete metric space and let (M_2, δ) any metric space. Then any function $f: M_1 \to M_2$ is continuous. ie any function whose domain is a discrete metric Space is continuous.

Proof:

Let $x \in M_1$ Let $\varepsilon > 0$ be given

Since M_1 is discrete for any $\delta < 1$,

$B(x, \delta) = \{x\}$.

$\therefore f(B(x, \delta)) = \{f(x)\} \subseteq B(f(x), \epsilon)$

$\therefore f$ is continuous at x.

We now give a characterization for continuity of a function at any terms of sequences converging to that point.

Theorem: 1

Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Let $a \in M_1$. A Function $f: M_1 \to M_2$ is continuous at a iff $(x_n) \to x \Rightarrow (f(x_n)) \to f(a)$.

Proof:

Suppose f is continuous at a

Let (x_n) be a sequence in M_1 such that $(x_n) \to a$

claim $(f(x_n)) \to f(a)$

Let $\varepsilon > 0$ be given.

there exists $\delta > 0$ such that
$d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \epsilon$ → ①

Since $(x_n) \to a$, there exists a positive integer n_0 such that
$d_1(x_n, a) < \delta \ \forall \ n \geq n_0$
$\therefore d_2(f(x_n), f(a)) < \epsilon \ \forall \ n \geq n_0$ (by 1)
$\therefore (f(x_n)) \to f(a)$

Conversely

Suppose $(x_n) \to a \Rightarrow f(x_n)) \to f(a)$
We claim that f is continuous at a
Suppose f is not continuous at a.
Then there exists an $\epsilon > 0$ such that
For all $\delta > 0$, $f(B(a, \delta)) \not\subset B(f(a), \epsilon)$
In particular $f(B(a, 1/n)) \not\subset B(f(a), \epsilon)$
Choose x_n such that
$x_n \in B(a, 1/n)$ and $f(x_n) \notin B(f(a), \epsilon)$
$\therefore d_1(x_n, a) < \frac{1}{n}$ and $d_2(f(x_n), f(a)) \geq \epsilon$
$\therefore (x_n) \to a$ and $(f(x_n))$ does not converge to $f(a)$ which is a contradiction to the hypothesis.
$\therefore f$ is continuous at a.

Theorem: 2

Let (M_1, d_1) and (M_2, d_2) be two metric spaces. A function $f: M_1 \to M_2$ is continuous iff $f^{-1}(G)$ is open in M_1 whenever G is open in M_2.

i.e. f is continuous iff inverse image of every open set is open.

Proof:

Suppose f is continuous.

Let G be an open set in M_2.

claim that $f^{-1}(G)$ is open in M_1.

If $f^{-1}(G)$ is empty, then it is open

Let $f^{-1}(G) \neq \phi$

Let $x \in f^{-1}(G)$. Hence $f(x) \in G$.

Since G is open, there exists an open ball $B(f(x), \epsilon)$ such that

$B(f(x), \epsilon) \subseteq G$. — ①

Now, by definition of continuity there exists an open ball $B(x, \delta)$ that $f(B(x, \delta)) \subseteq B(f(x), \epsilon)$

$\therefore f(B(x, \delta)) \subseteq G$ (by 1)

$\therefore B(x, \delta) \subseteq f^{-1}(G)$
since $x \in f^{-1}(G)$ is arbitrary. $f^{-1}(G)$ is open

Conversely, suppose $f^{-1}(G)$ is open in M_1 when ever G is open.

claim that f is continuous.

Let $x \in M_1$

Now $B(f(x), \epsilon)$ is an open set in M_2
$f^{-1}(B(f(x), \epsilon))$ is open in M_1 and
$f^{-1}(B(f(x), \epsilon))$.

There exists $\delta > 0$ such that
$B(x, \delta) \subseteq f^{-1}(B(f(x), \epsilon))$.

$\therefore f(B(x, \delta)) \subseteq B(f(x), \epsilon)$.

$\therefore f$ is continuous at x.

since $x \in M_1$ is arbitrary f is continuous.

Note 1. If $f : M_1 \to M_2$ is continuous and G is open in M_1, then it is not necessary that $f(G)$ is open in M_2.

ie) Under a continuous map the image of an open set need not be an open set.

For example let $M_1 = R$ with discrete metric and let $M_2 = R$ with usual metric.

Let $f : M_1 \to M_2$ be defined by $f(x) = x$

Since M_1 is discrete every subset of M_1 is open.

Hence for any open subset G of M_2, $f^{-1}(G)$ is open in M_1.

$\therefore$ f is continuous.

Now, $A = \{x\}$ is open in M_1.

But $f(A) = \{x\}$ is not open in M_2.

Note 2. In the above example f is a continuous bijection whereas $f^{-1} : M_2 \to M_1$ is not continuous.

For, $\{x\}$ is an open set in M_1.

$(f^{-1})^{-1}(\{x\}) = \{x\}$ which is not open in M_2.

$\therefore$ f^{-1} is not continuous.

Thus if f is a continuous bijection, f^{-1} need not be continuous

We now give yet another characterisation of continuous functions in terms of closed sets.

Theorem 4.3

Let (M_1, d_1) and (M_2, d_2) be two metric spaces. A function $f : M_1 \to M_2$ is continuous iff $f^{-1}(F)$ is closed in M_1 whenever F is closed in M_2

proof

Suppose $f : M_1 \to M_2$ is continuous.

Let $F \subseteq M_2$ be closed in M_2

$\therefore F^c$ is open in M_2.

$\therefore f^{-1}(F^c)$ is open in M_1.

But $f^{-1}(F^c) = [f^{-1}(F)]^c$.

$f^{-1}(F)$ is closed in M_1

conversely, Suppose $f^{-1}(F)$ is closed in M_1 whenever F is closed in M_2.

We claim that f is continuous.

Let G be an open set in M_2.

$\therefore G^c$ is closed in M_2.

$\therefore f^{-1}(G^c)$ is closed in M_1.

$\therefore [f^{-1}(G)]^c$ is closed in M_1.

$\therefore f^{-1}(G)$ is open in M_1.

$\therefore f$ is continuous.

We give one more characterisation of continuous function in terms of closure of a set.

<u>Theorem 4.4</u>

Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Then $f: M_1 \to M_2$ is continuous iff $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq M_1$.

proof

Suppose f is continuous.

Let $A \subseteq M_1$. then $f(A) \subseteq M_2$.

Since f is continuous, $f^{-1}(\overline{f(A)})$ is closed in M_1.

Also $f^{-1}(\overline{f(A)}) \supseteq A$ (since $\overline{f(A)} \supseteq f(A)$)

But $\overline{A}$ is the smallest closed set containing A.

$\therefore \overline{A} \subseteq f^{-1}(\overline{f(A)})$

$\therefore f(\overline{A}) \subseteq \overline{f(A)}$

conversely, let $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq M_1$.

To prove that f is continuous we shall show that if F is a closed set in M_2, then $f^{-1}(F)$ is closed in M_1.

By hypothesis, $f(\overline{f^{-1}(F)}) \subseteq \overline{f f^{-1}(F)}$

$\subseteq \overline{F}$.
$= F$ (since F is closed)

Thus $f(\overline{f^{-1}(F)}) \subseteq F$

$\therefore \overline{f^{-1}(F)} \subseteq f^{-1}(F)$

Also $f^{-1}(F) \subseteq \overline{f^{-1}(F)}$

$\therefore f^{-1}(F) = \overline{f^{-1}(F)}$

Hence $f^{-1}(F)$ is closed.

$\therefore f$ is continuous.

Solved problems

Problem 1

Let f be a continuous real valued function defined on a metric space M. Let $A = \{x \in M / f(x) \geq 0\}$. Prove that A is closed.

Solution

$$A = \{x \in M / f(x) \geq 0\}$$
$$= \{x \in M / f(x) \in [0, \infty)\}.$$
$$= f^{-1}([0, \infty)).$$

Also $[0, \infty)$ is a closed subset of R.

Since f is continuous $f^{-1}([0,\infty))$ is closed in M.

$\therefore A$ is closed.

Problem 2

Show that the function $f: R \to R$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$$

is not continuous by each of the following methods

(i) By the usual ϵ, δ method

(ii) By exhibiting a sequence (x_n) such that $(x_n) \to x$ and $(f(x_n))$ does not converge to $f(x)$.

(iii) By exhibiting an open set G such that $f^{-1}(G)$ is not open.

(iv) By exhibiting a closed subset F such that $f^{-1}(F)$ is not closed.

(v) By exhibiting a subset of A of R such that $f(\bar{A}) \not\subset \overline{f(A)}$

solution

(i) To prove that f is not continuous at x we have to show that there exists an $\varepsilon > 0$ such that for all $\delta > 0$, $f(B(x,\delta)) \not\subset B(f(x),\varepsilon)$.

Let $\varepsilon = \frac{1}{2}$

For any $\delta > 0$, $B(x,\delta) = (x-\delta, x+\delta)$ contains both rational and irrational numbers.

If x is rational, choose $y \in B(x,\delta)$ such that y is irrational and if x is irrational, choose $y \in B(x,\delta)$ such that y is rational.

Then $|f(x) - f(y)| = 1$ (by definition of f)

ie) $d(f(x), f(y)) = 1$

$\therefore f(y) \notin B(f(x), \frac{1}{2})$

Thus $y \in B(x,\delta)$ and $f(y) \notin B(f(x), \frac{1}{2})$

$\therefore f(B(x,\delta)) \not\subset B(f(x),\varepsilon)$.

Hence f is not continuous at x.

(ii) Let $x \in R$. Suppose x is rational. Then $f(x) = 1$

Let (x_n) be a sequence of irrational numbers such that $(x_n) \to x$.

Then $(f(x_n)) \to 0$ and $f(x) = 1$

$\therefore (f(x_n))$ does not converge to $f(x)$.

proof is similar if x is irrational

(iii) Let $G = (\frac{1}{2}, \frac{3}{2})$. Clearly G is open in R.

Now $f^{-1}(G) = \{x \in R \,/\, f(x) \in G\}$.

$= \{x \in R \,/\, f(x) \in (\frac{1}{2}, \frac{3}{2})\}$.

$= Q$

But Q is not open in R.

Thus $f^{-1}(G)$ is not open in R.

$\therefore f$ is not continuous.

(iv) choose $F = [1/2, 3/2]$

Then, $f^{-1}(F) = Q$ which is not closed in R.

$\therefore f$ is not continuous.

(v) Let $A = Q$. Then $\bar{A} = R$ (refer example 1(d) in 2.9)

$\therefore f(\bar{A}) = f(R) = \{0,1\}$ (by definition of f)

Also, $f(A) = f(Q) = \{1\}$.

$\therefore \overline{f(A)} = \overline{\{1\}} = \{1\}$

$f(\bar{A}) \not\subseteq \overline{f(A)}$

$\therefore f$ is not continuous.

Problem : 3

Let M_1, M_2, M_3 be metric spaces. If $f : M_1 \to M_2$ and $g : M_2 \to M_3$ are continuous functions, prove that $g \circ f : M_1 \to M_3$ is also continuous.

ie., composition of two continuous functions is continuous.

Solution:

Let G be open in M_3.

Since g is continuous, $g^{-1}(G)$ is open in M_2.

Now, since f is continuous, $f^{-1}(g^{-1}(G))$ is open in M_1.

ie., $(g \circ f)^{-1}(G)$ is open in M_1.

$\therefore$ $g \circ f$ is continuous.

Problem : 4

Let M be a metric space. Let $f : M \to R$ and $g : M \to R$ be two continuous functions. Prove that $f+g : M \to R$ is continuous.

Solution :

Let (x_n) be a sequence converging to x in M.

Since f and g are continuous functions, $(f(x_n)) \to f(x)$ and $(g(x_n)) \to g(x)$.

$\therefore \; (f(x_n) + g(x_n)) \to f(x) + g(x)$.

ie., $((f+g)(x_n)) \to (f+g)(x)$.

$\therefore \; f+g$ is continuous.

Problem : 5

Let f, g be continuous real valued functions on a metric space M. Let $A = \{x \,/\, x \in M \text{ and } f(x) < g(x)\}$. Prove that A is open.

Solution :

Since f and g are continuous real valued functions on M, $f-g$ is also a continuous real valued function on M.

Now, $A = \{x \in M \,/\, f(x) < g(x)\}$

$$= \{x \in M \,/\, f(x) - g(x) < 0\}$$

$$= \{x \in M \,/\, (f-g)(x) < 0\}$$

$$= \{x \in M \,/\, (f-g)x \in (-\infty, 0)\}$$

$$= (f-g)^{-1}\{(-\infty, 0)\}.$$

Now, $(-\infty, 0)$ is open in R, and $f-g$ is continuous.

Hence $(f-g)^{-1}\{(-\infty, 0)\}$ is open in M.

$\therefore$ A is open in M.

Problem : 6

If $f : R \to R$ and $g : R \to R$ are both continuous functions on R and if $h : R^2 \to R^2$ is defined by $h(x, y) = (f(x), g(y))$ prove that h is continuous on R^2.

Solution :

Let (x_n, y_n) be a sequence in R^2 converging to (x, y).

We claim that $(h(x_n, y_n))$ converges to $h(x, y)$.

Since $((x_n, y_n)) \to (x, y)$ in R^2, $(x_n) \to x$ and $(y_n) \to y$ in R.

Also f and g are continuous.

$\therefore (f(x_n)) \to f(x)$ and $g(y_n) \to g(y)$.

$\therefore (f(x_n), g(y_n)) \to (f(x), g(y))$.

$\therefore (h(x_n, y_n)) \to h(x, y)$.

$\therefore h$ is continuous on R^2.

Problem : 7

Let (M, d) be a metric space. Let $a \in M$. Show that the function $f : M \to R$ defined by $f(x) = d(x, a)$ is continuous.

Solution:

Let $x \in M$.

Let (x_n) be a sequence in M such that $(x_n) \to x$.

We claim that $(f(x_n)) \to f(x)$.

Let $\varepsilon > 0$ be given.

Now, $|f(x_n) - f(x)| = |d(x_n, a) - d(x, a)| \le d(x_n, x)$

Since $(x_n) \to x$, there exists a positive integer n_1 such that

$d(x_n, x) < \varepsilon$ for all $n \ge n_1$.

$\therefore$ $|f(x_n) - f(x)| < \varepsilon$ for all $n \ge n_1$.

$\therefore$ $(f(x_n)) \to f(x)$.

$\therefore$ f is continuous.

Problem : 8

Let f be a function from R^2 onto R defined by $f(x, y) = x$ for all $(x, y) \in R^2$. Show that f is continuous in R^2.

Solution :

Let $(x, y) \in R^2$.

Let $((x_n, y_n))$ be a sequence in R^2 converging to (x, y).

Then $(x_n) \to x$ and $(y_n) \to y$.

$\therefore$ $(f(x_n, y_n)) = (x_n) \to x = f(x, y)$

$\therefore$ $(f(x_n, y_n)) \to f(x, y)$.

$\therefore f$ is continuous.

Problem : 9

Define $f : l_2 \to l_2$ as follows. If $s \in l_2$ is the sequence $s_1, s_2, \ldots$ let $f(s)$ be the sequence $0, s_1, s_2, \ldots$ Show that f is continuous on l_2.

Solution:

Let $y = (y_1, y_2, \ldots, y_n \ldots) \in l_2$.

Let (x_n) be a sequence in l_2 converging to y.

Let $x_n = (x_{n1}, x_{n2}, \ldots, x_{nk}, \ldots)$

Then $(x_{n1}) \to y_1, (x_{n2}) \to y_2, \ldots (x_{nk}) \to y_k \cdots$

$\therefore (f(x_n)) = ((0, x_{n1}, x_{n2}, \ldots x_{nk}, \ldots) \to (0, y_1, y_2, \ldots y_k \ldots)$

$= f(y)$.

$\therefore (f(x_n)) \to f(y)$.

$\therefore f$ is continuous.

Problem : 10

Let G be an open subset of R. Prove that the characteristic function on G defined by $\chi_G(x) = \begin{cases} 1 & \text{if } x \in G \\ 0 & \text{if } x \notin G \end{cases}$

is continuous at every point of G.

Solution:

Let $x \in G$ so that $\chi_G(x) = 1$.

Let $\epsilon > 0$ be given.

Since G is open and $x \in G$, we can find a $\delta > 0$ such that $B(x, \delta) \subseteq G$.

$$\therefore \chi_G(B(x,\delta)) \subseteq \chi_G(G)$$
$$= \{1\}$$
$$\subseteq B(1, \epsilon)$$

Thus $\chi_G(B(x,\delta)) \subseteq B(\chi_G(x), \epsilon)$.

4:10 PM

Homomorphism

Definition:

Let (M_1, d_1) and (M_2, d_2) be metric spaces. A functions $f: M_1 \to M_2$ is called a homomorphism if

(i) f is 1-1 and onto.

ii) f is continuous.

iii) f^{-1} is continuous.

M_1 and M_2 are said to be homomorphic, if there exists a homomorphism $f: M_1 \to M_2$.

Definition:

A function $f: M_1 \to M_2$ is said to be an open map if $f(G)$ is open in M_1 for every open set G to M_1.

(i.e) f is an open map if the image of an open set in M_1 is an open set in M_1.

f is called a closed map if $f(P)$ is closed in M_2, for every closed set in M_2.

Note :1

Let $f: M_1 \to M_2$ be a 1-1 onto function. Then f^{-1} is continuous iff S is an open map.

For, f^{-1} is continuous iff for any open set G in M_1 M_1 $(f^{-1})^{-1}(G)$ is open in M_1.

$$\text{But } (f^{-1})^{-1}(G) = f(G).$$

$\therefore f^{-1}$ is continuous iff for every open set G in M_1, $f(G)$ is open in M_2.

$\therefore f^{-1}$ is continuous iff f is an open map.

Note : 2 ..

Similarly f^{-1} is continuous iff f is a closed map.

Note : 3

Let $f: M_1 \to M_2$ be a 1-1 onto map. Then the following are equivalent.

(i) f is a homomorphism.
(ii) f is a continuous open map
(iii) f is a continuous closed map.

Proof :-

(i) $\Leftrightarrow$ (ii) follows from note 1 and the definition of homomorphism.

(i) $\Leftrightarrow$ (iii) follows from note 2 and the definition of homomorphism.

Note : 4

Let $f: M_1 \to M_2$ be a homomorphism $G \subseteq M_1$ is open in M_1 iff $f(G)$ is open in M_2.

for, since f is an open map G is open in M_1 $\Big\}$ $\therefore f(G)$ is open in M_2.

Also, since f is continuous, $f(G)$ is open in M_2 $\Big\}$ $\therefore f^{-1}(f(G))$ is open in M_1.

$\therefore$ G is open in M_1, iff $f(G)$ is open in M_2 ——— (1)

conversely, if $f: M_1 \to M_2$ is a 1-1 onto map.

Satisfying (1) then f is a homomorphism.

Thus a homomorphism $f: M_1 \to M_2$ is simply 1-1, onto map between the points of the two spaces such that their open sets are also in 1-1 correspondence with each other.

Note : 5

Let $f: M_1 \to M_2$ be a 1-1 onto map. Then following conditions.

F is closed in M_1, iff $f(F)$ is closed in M_2.

Example :1

The metric space [0,1] and [0,2] with usual metric are homomorphic.

Proof:-

Define : $f : [0,1] \to [0,2]$ by $f(x) = 2x$

clearly f is 1-1 and onto.

Also $f^{-1}(x) = \frac{1}{2}x$

we note that f and f^{-1} are both continuous

$\therefore f$ is a homomorphism.

2) The metric space $(0,\infty)$ and R with usual metric are homomorphic.

Proof:-

$f : (0,\infty) \to R$ defined by $f(x) = \log_e x$ is the required homomorphism.

Hence $f^{-1}(x) = e^x$.

Ex:3

The metric spaces $(\frac{-\pi}{2}, \pi/2)$ and R with usual metric are homomorphic and $f : (\frac{-\pi}{2}, \pi/2) \to R$ defined by $f(x) = \tan x$ is the required homomorphic.

In example $(\frac{-\pi}{2}, \pi/2)$ is not a complete metric space whenever complete.

This shows that completeness of metric spaces is not presents homomorphism.

Ex:4 The metric spaces (0,1) and $(0,\infty)$ with usual metric homomorphism.

Proof:-

Define : $f : (0,1) \to (0,\infty)$ by $f(x) = \frac{x}{1-x}$

we claim that f is 1-1 and onto.

Let $f(x) = f(y)$.

$$\frac{x}{1-x} = \frac{y}{1-y}$$

$$x - xy = y - xy.$$

$x = y$, Hence f is 1-1.

Let $y \in (0, \infty)$

$\therefore f(x) = y \Rightarrow \frac{x}{1-x} = y$

$\Rightarrow x = y - xy$

$\Rightarrow y = x + xy$

$\Rightarrow y = x(1+y)$

$\Rightarrow x = y/1+y$

$\therefore \frac{y}{y+1} \in (0,1)$ is the preimage of y under f.

clearly f and f^{-1} are continuous.

$\therefore f$ is a homomorphism.

Ex:5 :-

R with usual metric is not homomorphic to R with discrete metric.

Proof :-

Let $M_1 = R$ with usual metric

Let $M_2 = R$ with discrete metric.

Let $f : M_1 \to M_2$ be any bijection.

Now, $\{a\}$ is open in M_2.

But $f^{-1}(\{a\}) = \{f^{-1}(a)\}$ is not open in M_1

Hence f is not continuous.

Thus any bijection $f : M_1 \to M_2$ is not a homomorphism.

Hence, M_1 is not homomorphic to M_2.

Definition:-

Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Let $f : M_1 \to M_2$ be a 1-1 onto map. f is said to be an isometry if $d_1(x, y) = d_2(f(x), f(y))$ for all $x, y \in M_1$. In other words, an isometry is a distance preserving map.

M_1 and M_2 are said to be isometric if there exists an isometry f from M_1 onto M_2.

Example 6:

R^2 with usual metric and C with usual metric are isometric and $f : R^2 \to C$ defined by $f(x, y) = x + iy$ is the required isometry.

Proof:-

Let d_1 denote the usual metric on R^2 and d_2 denote the usual metric on C

Let $a = (x_1, y_1)$ and $b = (x_2, y_2) \in R^2$

Then $d_1(a, b) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= |(x_1 - x_2) + i(y_1 - y_2)|$

$= |(x_1+iy_1)-(x_2+iy_2)|$

$= d_2(f(a), f(b))$

$\therefore$ f is an isometry.

Example 7:

Let d_1 be the usual metric on $[0,1]$ and d_2 be the usual metric on $[0,2]$.

The map $f:[0,1]\to[0,2]$ defined by $f(x)=2x$ is not an isometry.

Proof:

Let $x, y \in [0,1]$

Then $d_2(f(x), f(y)) = |f(x)-f(y)| = |2x-2y|$

$= 2|x-y| = 2d_1(x,y)$

$\therefore d_1(x,y) \neq d_2(f(x), f(y))$

Hence f is not an isometry.

Note:

Since an isometry f preserves distances, the image of an open ball $B(x,r)$ is the open ball $B(f(x), r)$.

Hence it follows that under an isometry

the image of an open set is also an open set.

Also if f is an isometry f^{-1} is also an isometry

Hence under an isometry the inverse image of an open set is open. Hence an isometry is a homeomorphism.

However a homeomorphism from one metric space to another need not be an isometry.

For example. $f : [0,1] \to [0,2]$ defined by $f(x) = 2x$ is a homeomorphism. (refer example 1)

But f is not an isometry (refer example 7)

Uniform continuity :-

Introduction:-

In this section, we introduce the concept of uniform continuity.

Let (M_1, d_1) and (M_2, d_2) be two metric spaces.

Let $f : M_1 \to M_2$ be a continuous function. For each $a \in M_1$ the following is true.

Given $\varepsilon > 0$, there exists $\delta > 0$ such that $d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \varepsilon$.

In general the number δ depends on ε. and the point a under consideration.

For example. consider $f : R \to R$ given by

$f(x) = x^2$.

Let $a \in R$. Let $\varepsilon > 0$ be given.

we want to find $\delta > 0$ such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

clearly. if $\delta > 0$ satisfies ①. then any δ_1. where $0 < \delta_1 < \delta$ also satisfies ①.

Hence if there exists a $\delta > 0$ satisfying ① then we can find another δ_1 such that $0 < \delta_1 < 1$ and δ_1 also satisfies ①.

Hence we may restrict x such that $|x-a| < 1$

$\therefore \quad a - 1 < x < a + 1$

$\therefore \quad x + a < 2a + 1$

$$\therefore \ |f(x) - f(a)| = |x^2 - a^2| = |x + a|\,|x - a|$$

$$< |2a + 1|\,|x - a| \text{ if } |x - a| < 1$$

Hence if we choose $\delta = \min \left\{ 1, \dfrac{\varepsilon}{|2a+1|} \right\}$ then

we have $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

Thus, in this example we see that the number δ depends on both ε and the point a under consideration and if a becomes large, δ has to be chosen correspondingly small. In fact, there is no $\delta > 0$ such that ① holds for all a.

For, suppose there exists $\delta > 0$ such that

$$|x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon \quad \text{for all } a \in R$$

Take $x = a + \frac{1}{2}\delta$

Clearly, $|x-a| = \frac{1}{2}\delta < \delta$

$$\therefore |f(x) - f(a)| < \varepsilon$$

$$\therefore \left| (a + \tfrac{1}{2}\delta)^2 - a^2 \right| < \varepsilon$$

$$\therefore \frac{1}{2}\delta \left| \frac{1}{2}\delta + 2a \right| < \varepsilon.$$

However, this inequality cannot be true for all $a \in R$, since by taking a sufficiently large we can make $\frac{1}{2}\delta \left| \frac{1}{2}\delta + 2a \right| > \varepsilon$.

Thus, there is no $\delta > 0$ such that ① holds for all $a \in R$.

We now consider another example.

Let $f: R \to R$ be given by $f(x) = 2x$

Let $a \in R$. Let $\varepsilon > 0$ be given

Then $|f(x) - f(a)| = |2x - 2a| = 2|x-a|$

$\therefore$ If we choose $\delta = \frac{1}{2}\varepsilon$ then we have

$$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Here δ depends on ε and not an a.

(i.e) for a given $\varepsilon > 0$ we are able to find $\delta > 0$ such that δ works uniformly for all $a \in R$.

Definition :- **Uniformly Continuous** :-

Let (M_1, d_1) and (M_2, d_2) be metric spaces. A function $f : M_1 \to M_2$ is said to be uniformly continuous on M_1 if given $\varepsilon > 0$ there exists $\delta > 0$ such that $d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon$.

Notes :-

1. Uniform continuity is global condition on the behaviour of a mapping on a set so that it is meaningless to ask whether a function is uniformly continuous at a point. Continuity is a local condition on the behaviour of a function at a point.

2. If $f : M_1 \to M_2$ is uniformly continuous on M_1, then f is continuous at every point of M_1. Moreover for a given $\varepsilon > 0$ there exists $\delta > 0$ such that $x, y \in M_1$. $d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon$.

Thus, uniform continuity is continuity plus the added conditions that for a given $\varepsilon > 0$ we can find $\delta > 0$ which works uniformly for all points of M_1.

3. A Continuous function $f : M_1 \to M_2$ need not be uniformly continuous on M_1.

For example, $f : R \to R$ defined by $f(x) = x^2$ is continuous but not uniformly continuous on R.

Solved Problems :

Problem 1 :

Prove that $f : [0,1] \to R$ defined by $f(x) = x^2$ is uniformly continuous on $[0,1]$.

Soln :

Let $\varepsilon > 0$ be given. Let $x, y \in [0,1]$

Then $|f(x) - f(y)| = |x^2 - y^2| = |x+y||x-y|$

$\leq 2|x-y|$

(since $x \leq 1$ and $y \leq 1$)

$\therefore |x-y| < \frac{1}{2}\varepsilon \Rightarrow |f(x) - f(y)| < \varepsilon$.

$\therefore f$ is uniformly continuous on $[0,1]$.

Problem 2 :

Prove that the function $f : (0,1) \to R$ defined by $f(x) = 1/x$ is not uniformly continuous.

Solution : Let $\varepsilon > 0$ be given. Suppose there exists $\delta > 0$ such that $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

Take $x = y + \frac{1}{2}\delta$.

Clearly $|x - y| = \frac{1}{2}\delta < \delta$.

$\therefore |f(x) - f(y)| < \varepsilon$.

$\therefore \left| \frac{1}{x} - \frac{1}{y} \right| < \varepsilon$.

$$\therefore \left| \frac{1}{y + \frac{1}{2}\delta} - \frac{1}{y} \right| < \varepsilon.$$

$$\therefore \left| \frac{\delta}{2\left(y + \frac{1}{2}\delta\right) y} \right| < \varepsilon.$$

$$\therefore \frac{\delta}{(2y + \delta)y} < \varepsilon.$$

This inequality cannot be true for all $y \in (0, 1)$ since $\frac{\delta}{(2y+\delta)y}$ becomes arbitrarily large as y approaches zero.

$\therefore f$ is not uniformly continuous.

Problem : 3

Prove that the function $f : R \to R$ defined by $f(x) = \sin x$ is uniformly continuous on R.

solution :

Let $x, y \in R$ and $x > y$

$$\sin x - \sin y = (x-y) \cos z$$

where $x > z > y$

(by mean value theorem)

$$\therefore |\sin x - \sin y| = |x-y| |\cos z|$$

$$\leq |x-y|$$

(since $|\cos x| \leq 1$).

Hence for a given $\varepsilon > 0$, if we choose $\delta = \varepsilon$, we have $|x-y| < \delta \Rightarrow |f(x) - f(y)| = |\sin x - \sin y| < \varepsilon$.

$\therefore f(x) = \sin x$ is uniformly continuous on R.

Discontinuous functions on R

In this section we shall investigate the set of points at which a given function $f : R \longrightarrow R$ is discontinuous. For this purupose the concept of the left limit and right limit of $f(x)$ at $x = a$ and classify the types of discontinuous for real functions. Thoroughout this section we deal with R with usual metric.

Definition:-

A function $f : R \longrightarrow R$ is said to approach to a limit l as x tends to a if given $\varepsilon > 0$ there exists $\delta > 0$ such that,

$0 < |x - a| < \delta \Rightarrow |f(x) - l| < \varepsilon$ and we write $\lim_{x \to a} f(x) = l$.

It should be carefully noted that the condition $0 < |x-a| < \delta$ excludes the point $x = a$ from consideration. Hence the definition of limit employs only values of x in some interval $(a - \delta, a + \delta)$ other than a. Hence the value of $f(x)$ at $x = a$ is immaterial and in fact to consider $\lim_{x \to a} f(x)$ the function $f(x)$ need not even be defined at $x = a$. Even if $f(a)$ is is defined it is not necessary that $\lim_{x \to a} f(x) = f(a)$.

When defining the limit of $f(x)$ as $x \longrightarrow a$ we consider the behaviour of $f(x)$ at points which are near to a and these points can be either to the left of a or to the right of a. However it is often necessary to know the behaviour of $f(x)$ as $x \longrightarrow a$ in such a way that x always remains greater than or less than a. This leads us to the concept of right and left limits of $f(x)$ at $x = a$.

Definition :-

A function f is that to have l as the right limit at $x = a$ if given $\varepsilon > 0$ there exists $\delta > 0$ such that $a < x < a + \delta \Rightarrow |f(x) - l| < \varepsilon$ and we write $\lim\limits_{x \to a^+} f(x) = l$.

Also we denote the right limit l by $f(a^+)$.

A function f is said to have l as the left limit at $x = a$ if given $\varepsilon > 0$ there exists $\delta > 0$ such that $a - \delta < x < a$ $\Rightarrow |f(x) - l| < \varepsilon$ and we write $\lim\limits_{x \to a^-} f(x) = l$.

Also we denote the left limit l by $f(a^-)$.

Note : 1

$\lim\limits_{x \to a} f(x) = l$ iff $\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^-} f(x) = l$.

(ie) $\lim\limits_{x \to a} f(x)$ exists iff the left and right limit of $f(x)$ at $x = a$ exists and are equal.

Note : 2

The definition of continuity of f at $x = a$ can be formulated as follows.

f is continuous at a iff $f(a^+) = f(a^-) = f(a)$

Note : 3

If $\lim\limits_{x \to a} f(x)$ does not exists then one of the following happens.

(i) $\lim\limits_{x \to a^+} f(x)$ does not exist.

(ii) $\lim\limits_{x \to a^-} f(x)$ does not exist.

(iii) $\lim\limits_{x \to a^+} f(x)$ and $\lim\limits_{x \to a^-} f(x)$ exist and are unequal.

Definition :-

If a function is discontinuous at a then a is called a point of discontinuity for the function.

If a is a point of discontinuity of a function then any one of the following cases arises.

(i) $\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$.

(ii) $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist and are not equal.

(iii) Either $\lim_{x \to a^-} f(x)$ and or $\lim_{x \to a^+}$ does not exist.

Definition :-

Let a be a point of discontinuity for $f(x)$. a is said to be a point of discontinuity of the first kind if $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist and both of them are finite and unequal.

a is said to be a point of discontinuity of the second kind if either $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ does not exist.

Definition :-

Let $A \subseteq R$. A function $f : A \to R$ is called monotonic increasing if $x, y \in A$ and $x < y \Rightarrow f(x) \leq f(y)$.

f is called monotonic decreasing if $x, y \in A$ and $x > y \Rightarrow f(x) \geq f(y)$.

f is called monotonic if it is either monotonic increasing or monotonic decreasing.

Theorem 4.6

Let $f : [a,b] \to R$ be a monotonic increasing function. Then f has a left limit at every point of (a,b). Also f has a right limit at a and has a left limit at b. Further $x < y \Rightarrow f(x+) \leq f(y-)$ Similiar result is true for mono-tonic decreasing functions.

Proof:

Let $f : [a,b] \to R$ be monotonic increasing. Let $x \in [a,b]$. Then $\{f(t) : a \leq t < x\}$ is bounded above by $f(x)$

Let $l = l.u.b \{f(t) : a \leq t < x\}$

We claim that $f(x-) = l$

Let $\varepsilon > 0$ be given. By definition of l.u.b there exists t such that $a \leq t < x$ and $l - \varepsilon < f(t) \leq l$

$\therefore t < u < x \Rightarrow l - \varepsilon < f(t) \leq f(u) \leq l$

$\Rightarrow l - \varepsilon < f(u) \leq l$ (Since f monotonic increasing)

$x - \delta < u < x \Rightarrow l - \varepsilon < f(u) \leq l$ where $\delta = x - t$

$\therefore f(x-) = l$

Similarly we can prove that $f(x+) = g.l.b \{f(t) : x < t \leq b\}$

Now, we shall prove that $x<y \Rightarrow f(x+) \leq f(y-)$

Let $x<y$

Now $f(x+) = g.l.b \{f(t): x<t \leq b\}$

$= g.l.b \{f(t): x<t \leq y\}$ (Since f is monotonic increasing) ①

Also $f(y-) = l.u.b \{f(t): a \leq t < y\}$

$= l.u.b \{f(t): x \leq t < y\}$ — ②

It follows from (1) and (2) that $f(x+) \leq f(y-)$

The proof for monotonic decreasing function is similiar.

Theorem 4 6

Let $f: [a,b] \to R$ be a monotonic function. Then the set of points of $[a,b]$ at which f is discontinuous is countable.

Proof:

we shall prove that the theorem for a monotonic increasing function

Let $E = \{x / x \in [a,b]$ and f is discontinuous at $x\}$

Let $x \in E$. Then by theorem 4.5,

$f(x+)$ and $f(x-)$ exist and $f(x-) \leq f(x) \leq f(x+)$

If $f(x-) = f(x+)$ then $f(x-) = f(x) = f(x+)$

$\therefore$ f is continuous at x which is a contradiction

$\therefore f(x-) \neq f(x+)$

$\therefore f(x-) < f(x+)$

Now choose a rational number $r(x)$ such that $f(x-) < r(x) < f(x+)$. This defines a map r from E to Q which maps x to $r(x)$

We claim that r is 1-1

Let $x_1 < x_2$

$\therefore f(x_1+) < f(x_2-)$ (by theorem 4.5)

Also $f(x_1-) < r(x_1) < f(x_1+)$ and

$f(x_2-) < r(x_2) < f(x_2+)$

$\therefore r(x_1) < f(x_1+) < f(x_2-) < r(x_2)$.

Thus $x_1 < x_2 \Rightarrow r(x_1) < r(x_2)$

$\therefore r : E \to Q$ is 1-1. Hence E is countatable.

Thus we have proved that the set of discontinuities of a monotonic function is countable. We now proceed to investigate the nature of the set of discontinuties of any real valued function.

<u>Definition:</u>

A subset D of R is said to be of type F_σ if D can be expressed as a countable union of closed sets (ie) $D = \cup F_n$ where every F_n is a closed subset of R.

Note 1:

Any closed subset F is of type F_σ since $F = \bigcup_{n=1}^{\infty} F_n$ where $F_n = F$ for all n.

Note 2:

A set of type F_σ need not be closed

For example any countable set D is of type F_σ since it can be expressed as a countable union of singleton sets. Thus $\mathbb{Q}$ is of type F_σ. However $\mathbb{Q}$ is not closed.

We now proceed to prove that the set of discontinuities of a function $f: R \to R$ is of type F_σ.

Definition:

Consider any function $f: R \to R$ Let I be a bounded open interval in R. Then the oscillation of f over I denoted by $\omega(f, I)$ is defined by $\omega(f, I) = l.u.b \{f(x) / x \in I\} - g.l.b \{f(x) / x \in I\}$.

If $a \in R$ the oscillation of f at a denoted by $\omega(f, a)$ is defined by $\omega(f, a) = g.l.b\ \omega(f, I)$ where g.l.b is taken over all bounded open intervals containing a.

Example :-

consider the function $f: R \to R$ defined by $f(x) = [x]$

Let $a = 4$. Let I be any bounded open interval containing 4.

Suppose I does not contain any integer other than 4. Then $\omega(f, I) = 4 - 3 = 1$

For any other open interval I containing 4, $\omega(f, I) \geq 1$

$\therefore \omega(f, 4) = 1$

In general for any $n \in Z$, $\omega(f, n) = 1$

Note:-

It follows from the definition that $\omega(f, I) \geq 0$ for any I. Hence $\omega(f, a) \geq 0$ for any point $a \in R$.

Theorem 4.7.

$f: R \to R$ is continuous at $a \in R$ iff $\omega(f,a) = 0$.

proof:

Suppose f is continuous at a.

Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that

$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \frac{1}{2}\varepsilon$.

let $I = (a-\delta, a+\delta)$

For any $x \in I$, $|f(x) - f(a)| < \frac{1}{2}$.

For any $x, y \in I$ $|f(x) - f(y)| \leq |f(x) - f(a)| + |f(a) - f(y)|$

$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$.

$\therefore \omega(f, I) < \varepsilon$.

since $\varepsilon > 0$ is arbitrary $\omega(f,a) = 0$.

conversely, let $\omega(f,a) = 0$, we claim that f is continuous at a.

let $\varepsilon > 0$ be given.

since $\omega(f,a) = g.l.b\ \omega(f, I) = 0$ there exists a bounded open interval I containing a such that $\omega(f, I) < \varepsilon$ — ①

Let $x_1, x_2 \in I$.

Then $f(x_1) \leq l.u.b \{f(x) / x \in I\}$.

$f(x_2) \geq g.l.b \{f(x) / x \in I\}$.

$|f(x_1) - f(x_2)| \leq \omega(f, I) < \varepsilon$ [by ①]

Thus for any two points $x_1, x_2 \in I$, $|f(x_1) - f(x_2)| < \varepsilon$.

In particular $|f(x) - f(a)| < \varepsilon$ for all $x \in I$.

Now, since I is a bounded open interval containing a we can choose $\delta > 0$ such that $(a-\delta, a+\delta) \subseteq I$

$|f(x) - f(a)| < \varepsilon$ for all $x \in (a-\delta, a+\delta)$.

$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

f is continuous at a.

Theorem 4.8.

Let $f : R \to R$ be any function let $r > 0$ Then $E_r = \{a \in R \mid \omega(f, a) \geq \frac{1}{r}\}$ is a closed set.

proof:

Let x be any limit point of E_r.

we claim that $x \in E_r$.

For this we must prove that $\omega(f, x) \geq 1/r$.

Now, let I be any bounded open interval containing x.

Since x is a limit point of E_r, I contains a point y of E_r.

Hence I is a bounded open interval containing y.

$\omega(f, y) \leq \omega(f, I)$.

But $\omega(f, y) \geq \frac{1}{r}$ (since $y \in E_r$)

$\omega(f, I) \geq 1/r$ and this is true for any bounded open interval I containing x.

$\omega(f, x) \geq 1/r$.

$\therefore x \in E_r$.

$\therefore E_r$ contains all its limit point.

E_r is closed.

Theorem 4.9.

Let D be the set of point of discontinuities of a function $f: R \to R$. Then D is of type F_σ.

proof:

Let $x \in D$. Then f is continuous at x.

$\omega(f, x) > 0$.

$\omega(f, x) \geq 1/n$ for some positive integer n.

$x \in E_n$ for some positive integer n where E_n is defined as in $x \in \bigcup_{n=1}^{\infty} E_n$.

$$D \subseteq \bigcup_{n=1}^{\infty} E_n \quad \text{——} \quad (1).$$

Now let $x \in \bigcup_{n=1}^{\infty} E_n$.

Then $x \in E_n$ for some positive integer n.

$\omega(f, x) \geq 1/n$. Hence $\omega(f, x) > 0$

f is discontinuous at x. Hence $x \in D$.

$$\bigcup_{n=1}^{\infty} E_n \subseteq D \quad \text{——} \quad (2).$$

Thus $D = \bigcup_{n=1}^{\infty} E_n$ [by (1) and (2)]

Also each E_n is closed.

Thus D is countable union of closed sets.

D is of type F_σ.

Theorem 4.10.

There is no function $f: R \to R$ such that f is continuous at each rational number and discontinuous at each irrational number.

proof:

The set A of all irrational numbers is not of type F_σ.

suppose A is of type F_σ.

Then $A = \bigcup_{n=1}^{\infty} F_n$ where each F_n is closed.

Now, since F_n contains only irrational numbers. F_n cannot contain any open interval.

$\text{Int } F_n = \phi$.

$\text{Int } \overline{F_n} = \phi$ (since F_n is closed)

F_n is nowhere dense.

A is of first category which is a contradiction.

A is not of type F_σ.

Hence the theorem.